Data for Section 7.1

25 most populous US states
Source: US Census Bureau (2010 Census)

<table>
<thead>
<tr>
<th>State</th>
<th>Population (millions)</th>
<th>Rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>California</td>
<td>37.3</td>
<td>1</td>
</tr>
<tr>
<td>Texas</td>
<td>25.1</td>
<td>2</td>
</tr>
<tr>
<td>New York</td>
<td>19.4</td>
<td>3</td>
</tr>
<tr>
<td>Florida</td>
<td>18.8</td>
<td>4</td>
</tr>
<tr>
<td>Illinois</td>
<td>12.8</td>
<td>5</td>
</tr>
<tr>
<td>Pennsylvania</td>
<td>12.7</td>
<td>6</td>
</tr>
<tr>
<td>Ohio</td>
<td>11.5</td>
<td>7</td>
</tr>
<tr>
<td>Michigan</td>
<td>9.9</td>
<td>8</td>
</tr>
<tr>
<td>Georgia</td>
<td>9.7</td>
<td>9</td>
</tr>
<tr>
<td>North Carolina</td>
<td>9.5</td>
<td>10</td>
</tr>
<tr>
<td>New Jersey</td>
<td>8.8</td>
<td>11</td>
</tr>
<tr>
<td>Virginia</td>
<td>8.0</td>
<td>12</td>
</tr>
<tr>
<td>Washington</td>
<td>6.7</td>
<td>13</td>
</tr>
<tr>
<td>Massachusetts</td>
<td>6.5</td>
<td>14</td>
</tr>
<tr>
<td>Indiana</td>
<td>6.5</td>
<td>15</td>
</tr>
<tr>
<td>Arizona</td>
<td>6.4</td>
<td>16</td>
</tr>
<tr>
<td>Tennessee</td>
<td>6.3</td>
<td>17</td>
</tr>
<tr>
<td>Missouri</td>
<td>6.0</td>
<td>18</td>
</tr>
<tr>
<td>Maryland</td>
<td>5.8</td>
<td>19</td>
</tr>
<tr>
<td>Wisconsin</td>
<td>5.7</td>
<td>20</td>
</tr>
<tr>
<td>Minnesota</td>
<td>5.3</td>
<td>21</td>
</tr>
<tr>
<td>Colorado</td>
<td>5.0</td>
<td>22</td>
</tr>
<tr>
<td>Alabama</td>
<td>4.8</td>
<td>23</td>
</tr>
<tr>
<td>South Carolina</td>
<td>4.6</td>
<td>24</td>
</tr>
<tr>
<td>Louisiana</td>
<td>4.5</td>
<td>25</td>
</tr>
</tbody>
</table>

Postal codes of 50 US states (alphabetized by state name)
Source: US Postal Service

| AL | AK | AZ | AR | CA | CO | CT | DE | FL | GA | HI | ID | IL | IN | IA | KS | KY | LA | MD | ME | MA | MI | MN | MS | MO | MT | NE | NV | NH | NJ | NM | NY | NC | ND | OH | OK | OR | PA | RI | SC | SD | TN | TX | UT | VT | VA | WA | WV | WI | WY |
The Real Number System

Real numbers

<table>
<thead>
<tr>
<th>Rational numbers</th>
<th>Irrational numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{2}{3} ), ( -\frac{5}{7} ), ( \frac{13}{5} ), 5.2</td>
<td>( \sqrt{2}, \sqrt{6}, \sqrt{7}, \pi )</td>
</tr>
</tbody>
</table>

Integers: \( \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \)

Whole numbers: \( \{0, 1, 2, 3, 4, 5, \ldots\} \)

Counting numbers or Natural numbers: \( N = \{1, 2, 3, 4, 5, \ldots\} \)

Whole Numbers: \( W = \{0, 1, 2, 3, 4, 5, \ldots\} \)

Integers: \( Z = \{\ldots -4, -3, -2, -1, 0, 1, 2, 3, \ldots\} \)

- The integers consist of the set of whole numbers and their opposites.

Rational Numbers: \( Q = \left\{ \frac{a}{b} \mid a \in Z \text{ and } b \in Z \text{ and } b \neq 0 \right\} \)

- The set of rational numbers consists of all numbers that can be written as the ratio of two integers (with a non-zero denominator), i.e., the set of all numbers that can be written as fractions. This includes integers, terminating decimals, repeating decimals as well as fractions.

Irrational Numbers: \( \{x \mid x \in \mathbb{R} \text{ and } x \notin Q \} \)

- The set of irrational numbers consists of all real numbers that are not rational.

Real Numbers: \( \mathbb{R} \)

- The real numbers refers to the set of all numbers that can be represented on the number line.

NOTE:
- \( W = N \cup \{0\} \)
- The rational numbers and the irrational numbers are two disjoint sets.
- Each of the sets described above is a subset of \( \mathbb{R} \).
Sets

A set is a well-defined collection of objects

- The objects are called elements or members of the set.
- “Well defined” means you can clearly tell whether or not an object is included in the set.

EXAMPLE: Sets

<table>
<thead>
<tr>
<th>Not a set (not well-defined)</th>
<th>Set (well-defined collection of objects)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Large US states</td>
<td>US states whose population is greater than 15 million.</td>
</tr>
<tr>
<td>&quot;Large&quot; is not well-defined.</td>
<td>We can clearly tell whether a state has a population greater than 15 million (in which case it is a member of the set) or does not (in which case it is not a member).</td>
</tr>
<tr>
<td>Tall people</td>
<td>People who are 6’2” or taller</td>
</tr>
</tbody>
</table>

Describe the set of states whose population is greater than 15 million:

Roster Notation

There are several ways to describe sets. One of the easiest ways is roster notation, where you list the elements of the set between braces, separated by commas.

- You may use an ellipsis (three dots) to show that a pattern continues.

EXAMPLE: Roster Notation

Describe each set using roster notation.

<table>
<thead>
<tr>
<th>Set</th>
<th>Number of elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>US states whose population is greater than 15 million.</td>
<td></td>
</tr>
<tr>
<td>Days of the week.</td>
<td></td>
</tr>
<tr>
<td>Natural numbers less than 6</td>
<td></td>
</tr>
<tr>
<td>Natural numbers less than 100</td>
<td></td>
</tr>
<tr>
<td>Natural numbers that are multiples of 5</td>
<td></td>
</tr>
</tbody>
</table>
Set Names

We typically use capital letters to name sets.

EXAMPLE: Set Names

- $A = \{\text{CA, TX, NY, FL}\}$
- $B = \{\text{Mon., Tues., Wed., Thu., Fri., Sat., Sun.}\}$
- $C = \{1, 2, 3, 4, 5\}$

The Symbols $\in$ and $\notin$

Because sets are well-defined, an object is either a member or not a member of a set.

- $3 \in C$ means “3 is a member of set $C$."
- $0 \notin C$ means “0 is not a member of set $C$."
- $\text{IL} \notin \{\text{CA, TX, NY, FL}\}$

Some frequently used sets have standard names (e.g., see page 2).

EXAMPLE: Sets

Is the following a set, i.e., is it a well-defined collection of objects?

US states whose names begin with "Z"
The Empty Set (or Null Set)

The **empty set** (also called the **null set**) is a set that contains no elements.

**EXAMPLE: The Empty Set**
- US states whose names begin with "Z"
- Months with 40 days
- Natural numbers between 1 and 2

\(\emptyset\) is used to represent the empty set.
- \{\} is a less common way of representing the empty set.
- \{\emptyset\} is not the empty set; it is a set that contains *one element* (that element is the empty set).
- \{0\} is not the empty set; it is a set that contains *one element* (that element is the number zero).

**Set-Builder Notation**

- We can use roster notation to describe the set of *natural numbers* between 3 and 7, inclusive, namely: \{3, 4, 5, 6, 7\}.
- We cannot use roster notation to describe the set of *real numbers* between 3 and 7, inclusive.

Therefore, we need another way to describe sets.

**Set-builder notation** describes sets using a “membership rule” or property that describes the elements in the set. Set builder notation is particularly useful when it is impossible or inconvenient to list the elements using roster notation.

**Membership property or rule:**
- If \(x\) satisfies the rule, then \(x\) is a member of the set.
- If \(x\) does not satisfy the rule, then \(x\) is not a member of the set.

\[ \{ x \mid 3 \leq x \leq 7 \} \]

The set of all elements \(x\) such that \(x\) is between 3 and 7, inclusive.

**EXAMPLE: Set-Builder Notation**
Which of the following are elements of the set described above?

\[ 4.5 \quad 8 \quad -4 \]
### Roster Notation vs. Set-Builder Notation

Notice that both roster notation and set-builder notation use braces.

<table>
<thead>
<tr>
<th>Roster notation</th>
<th>Set builder notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>List of elements</td>
<td>{x \mid x \text{ has property } P}</td>
</tr>
<tr>
<td>{CA, TX, NY, FL}</td>
<td>{x \mid x \text{ is a state whose pop. } &gt; 15 \text{ million}}</td>
</tr>
<tr>
<td>{5, 10, 15, 20, 25, ...}</td>
<td>{x \mid x \text{ is a natural number and a multiple of } 5}</td>
</tr>
<tr>
<td>{3, 4, 5, 6, 7}</td>
<td>{x \mid x \text{ is a natural number and } 3 \leq x \leq 7}</td>
</tr>
<tr>
<td>No roster notation</td>
<td>{x \mid x \text{ is a real number and } 3 \leq x \leq 7}</td>
</tr>
<tr>
<td>{x \mid x \text{ is a real number and } 3 \leq x \leq 7}</td>
<td></td>
</tr>
</tbody>
</table>

For some sets, either notation may be used, but one may be more convenient. For example, it would be easier to write

\[ \{x \mid x \text{ is a US state whose pop. } \leq 15 \text{ million}\} \]

than to list all 46 members of this set using roster notation.

### The Cardinal Number of a Set

The **cardinal number** of set \( A \), written \( n(A) \), is the number of unique elements in set \( A \).

**EXAMPLE: Cardinal Number**

Let \( A = \{CA, TX, NY, FL\} \).

- \( n(A) = \)

**EXAMPLE: Cardinal Number**

Find the cardinal number of these sets:

<table>
<thead>
<tr>
<th>Set description (English)</th>
<th>Set description (Roster Notation)</th>
<th>Cardinal Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Natural numbers less than 100</td>
<td>(D = {1, 2, 3, 4, \ldots, 99})</td>
<td></td>
</tr>
<tr>
<td>Natural numbers that are multiples of 5</td>
<td>(E = {5, 10, 15, 20, 25, \ldots})</td>
<td></td>
</tr>
<tr>
<td>Natural numbers between 3 and 7 inclusive</td>
<td>(F = {3, 4, 5, 6, 7})</td>
<td></td>
</tr>
<tr>
<td>Real numbers between 3 and 7, inclusive</td>
<td>(G = {x \mid 3 \leq x \leq 7})</td>
<td></td>
</tr>
</tbody>
</table>
EXAMPLE: Cardinal Number (cont’d)
Find the cardinal number of these sets:

<table>
<thead>
<tr>
<th>Set description (English)</th>
<th>Set description (Roster Notation)</th>
<th>Cardinal Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Natural numbers between 1 and 2, exclusive</td>
<td>$H = {x</td>
<td>x \in N \text{ and } 1 &lt; x &lt; 2}$</td>
</tr>
<tr>
<td>A set consisting of the number 0.</td>
<td>$K = {0}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$L = {7, 8, 8, 9, 9, 9}$</td>
<td></td>
</tr>
</tbody>
</table>

Set Equality

Sets that are composed of the same elements are equal.
- It does not matter how the elements are listed or whether some of the elements are listed more than once!

EXAMPLE: Set Equality
The following sets are equal:

$$\{7, 8, 9\} = \{8, 9, 7\} = \{7, 9, 8\} = \{7, 8, 8, 9, 9, 9\}$$

The cardinal number of these sets is _______.

NOTE: Equal sets are composed of the same elements and therefore have the same cardinal number.

Set Inequality

Sets that are not composed of the same elements are not equal.

EXAMPLE: Set Inequality
The following sets are not equal:
- $\{7, 8, 9\} \neq \{8, 9, 10\}$
- $\{7, 8, 9\} \neq \{6, 7, 8\}$
- $\{7, 8, 9\} \neq \{0, 7, 8, 9\}$
- $\{7, 8, 9\} \neq \{7, 9\}$
Subsets and Proper Subsets

\[ A \subseteq B \]  
Set \( A \) is a subset of \( B \) if and only if every element of \( A \) is also an element of \( B \).

\[ A \subset B \]  
Set \( A \) is a proper subset of \( B \) if and only if:
- \( A \) is a subset of \( B \), and
- \( A \) is not equal to \( B \) (i.e., there is at least one element in \( B \) that is not an element of \( A \)).

In other words, a proper subset is a subset that is not the entire set.

NOTE: A proper subset is a type of subset. In other words, proper subsets are also subsets.

If \( A \subseteq B \), then \( A \subseteq B \).

EXAMPLE: Subsets and Proper Subsets
Let \( A = \{7,8,9\} \), \( B = \{6,7,8\} \), \( C = \{0,7,8,9\} \), \( D = \{7,8\} \), \( E = \{8,9,7\} \)

- List all sets shown above that are subsets of \( C \).
- List all sets shown above that are proper subsets of \( C \).
- List all sets shown above that are subsets of \( A \).
- List all sets shown above that are proper subsets of \( A \).
- Is \( \emptyset \) a subset of \( A \)?
Subsets and Proper Subsets (cont’d)

EXAMPLE: Subsets and Proper Subsets
Let \( M \) = male students in our class, \( C \) = all students in our class. Decide whether each of the following statements is true.

- \( M \subseteq C \)
- \( C \subseteq M \)

The following properties about sets and subsets are derived directly from the definitions:

- Every set is a subset (but not a proper subset) of itself.
  
  For all sets \( A \), \( A \subseteq A \)

- The empty set is a subset of every set.
  
  For all sets \( A \), \( \emptyset \subseteq A \)

- Two sets are equal if and only if they are subsets of each other.
  
  \( A = B \) if and only if \( A \subseteq B \) and \( B \subseteq A \).
Venn Diagrams

Venn Diagrams show the relationship between sets (and subsets). Sets are represented by circles, drawn within a rectangle representing the universal set \( U \).

This Venn Diagram shows that \( A \subseteq B \):

Do not read anything else into the diagram, e.g., do **not** assume that:
- \( A \) is a *proper* subset of \( B \)
- \( A \) is *small* relative to \( B \)
- \( B \) is *small* relative to the universal set \( U \).

It is even possible that \( A = B \) (in other words, there may be no elements in the portion of circle \( B \) outside of circle \( A \)).

However, if we *know* that two sets are equal, we can represent the two sets as a single circle.

Venn Diagrams can show the relationship among more than two sets, e.g., in this diagram: \( A \subseteq B \subseteq C \)

**Universal Set**

The **universal set** \( U \) is the set that contains all the elements being discussed. In a Venn Diagram, the universal set consists of the *entire* rectangle (*not* just the part of the rectangle outside the circles). All sets are subsets of the universal set.

Sometimes the universal set will be defined explicitly, and sometimes it will be obvious from the discussion.
Counting the Number of Subsets

Find the number of subsets for each of the following sets:

- $\{5\}$
- $\{7,8\}$
- $\{a,b,c\}$

A set with $n$ elements has $2^n$ subsets, including the empty set and the entire set.

EXAMPLE: Counting the Number of Subsets
Find the number of subsets for each of the following sets:

- Set of Days of the Week:

- Empty set $\emptyset$:

EXAMPLE: Pizza Toppings
A restaurant offers six different toppings on its pizzas (in addition to cheese). How many different ways are there to order a pizza if you can order any number of toppings, from zero to all six?
Set Operations

When we perform operations (e.g., addition, subtraction, multiplication, division) on real numbers, the result is a real number. We can also perform operations on sets. We will discuss three set operations in this course: complement, intersection and union. When we perform operations on sets, the result is a set.

Complement

The complement of set $A$ (written $A'$) is the set of elements in the universal set that are not in set $A$.

$$A' = \{ x | x \in U \text{ and } x \notin A \}$$

EXAMPLES: Complement

Describe the complement of each set.

<table>
<thead>
<tr>
<th>Set $A$</th>
<th>Complement</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U = {1, 2, 3, 4, \ldots, 11}$</td>
<td>$A'$</td>
</tr>
<tr>
<td>$A = {1, 2, 4, 5, 7}$</td>
<td></td>
</tr>
<tr>
<td>$U =$ Set of US states</td>
<td>$A'$</td>
</tr>
<tr>
<td>$A =$ Set of US states whose population is greater than 15 million</td>
<td>$n(A') =$</td>
</tr>
<tr>
<td>$U =$ Set of students in this class</td>
<td>$M'$</td>
</tr>
<tr>
<td>$M =$ Set of male students in this class</td>
<td></td>
</tr>
</tbody>
</table>

EXAMPLES: Complement

Describe each of the following:

- $\emptyset '$ (complement of the empty set)
- $U '$ (complement of the universal set)
- $(A')'$ (complement of the complement of $A$)
Intersection

The intersection of two sets $A$ and $B$ (written $A \cap B$) is the set of all elements that are common to both $A$ and $B$. In other words, $A \cap B$ is where sets $A$ and $B$ overlap.

$$A \cap B = \{x | x \in A \text{ and } x \in B\}$$

EXAMPLES: Intersection and Complement

Let $U = \{1, 2, 3, 4, \ldots, 11\}$, $A = \{1, 2, 4, 5, 7\}$, $B = \{2, 4, 5, 7, 9, 11\}$

- Find $A \cap B$

- Find $A \cap B'$

- Find $(A \cap B)'$

EXAMPLES: Intersection and Complement

Let $U = \text{Set of all US states}$.

- $A = \text{Set of US states whose population is greater than 15 million}$
- $C = \text{Set of US states whose names begin with “C”}$

- Find $A \cap C$

- Find $A \cap C'$
EXAMPLE: Intersection and Complement
Let \( U \) = Set of students in this class.
\( M \) = Set of male students in this class.
\( C \) = Set of students in this class who live in Cicero.

- Describe \( M \cap C \) in English.

EXAMPLE: Intersection and Complement
Let \( U \) = Set of natural numbers
\( A \) = Set of odd natural numbers
\( B \) = Set of even natural numbers

- Find \( A \cap B \)

**Disjoint Sets**

Two sets are **disjoint** if and only if they have no elements in common, i.e., \( A \) and \( B \) are disjoint if and only if \( A \cap B = \emptyset \).

When we know two sets are disjoint, we can draw a Venn Diagram with two non-intersecting circles.
Union

The **union** of two sets $A$ and $B$ (written $A \cup B$) is the set of all elements that are in $A$ or $B$ (or both). In other words, we form $A \cup B$ by combining the two sets $A$ and $B$.

$$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$$

**EXAMPLES: Union and Complement**

Let $U = \{1,2,3,4,\ldots,11\}$, $A = \{1,2,4,5,7\}$, $B = \{2,4,5,7,9,11\}$

- Find $A \cup B$

- Find $A \cup B'$

- Find $(A \cup B)'$

**EXAMPLES: Union and Complement**

Let $U =$ Set of all US states.

$A =$ Set of US states whose population is greater than 15 million

$C =$ Set of US states whose names begin with “C”

- Find $A \cup C$
EXAMPLE: Union and Complement
Let $U$=Set of students in this class.
$M$=Set of male students in this class.
$C$=Set of students in this class who live in Cicero.

- Describe $M \cup C$ in English.

EXAMPLE: Union and Complement
Let $U$=Set of natural numbers
$A$=Set of odd natural numbers
$B$=Set of even natural numbers

- Find $A \cup B$

EXAMPLES: Unions and Intersections of Complements
Describe each of the following. Refer to the diagram if necessary.

- $A \cup A'$

- $A \cap A'$
EXAMPLES: Set Operations

Let

\[ U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \]
\[ D = \{0, 1, 2, 3, 4, 5\} \]
\[ E = \{1, 3, 5, 7, 9\} \]
\[ F = \{2, 4, 5, 7, 9\} \]
\[ G = \{0, 6, 8\} \]

Find each of the following:

1. \((D \cap E) \cup F'\)
2. \((E \cup F)' \cap D\)
3. \(D \cap G\)
4. \(E \cap G\)
EXAMPLES: Set Operations
Let

\[ U = \text{Set of students in this class} \]
\[ M = \text{Set of male students (in this class)} \]
\[ E = \text{Set of Morton East graduates (in this class)} \]
\[ J = \text{Set of students with a full-time job (in this class)} \]

Describe each of the following in English. Use a Venn Diagram if you find it helpful.

- \( M \cap E \cap J \)
- \( M' \cap J \)
- \( M \cup E' \)
- \( (E \cup J)' \)

Assume Miguel is a student in this class who is male, a Morton West graduate, and does not have a full-time job. Determine whether Miguel is or is not a member of the following sets.

- \( M \cap E \cap J \)
- \( M' \cap J \)
- \( M \cup E' \)
- \( (E \cup J)' \)